# Complex difference system models for asymmetric interaction

Naohito Chino

**Abstract** Complex difference system models for asymmetric interaction addressed here were first proposed by the author at The International Conference on Measurement and Multivariate Analysis held on May, 2000, in Banff, Canada, which was organized by Sizuhiko Nishisato. In general, asymmetric interactions among members can be observed as a set of longitudinal asymmetric similarity matrices. Traditionally, these data have been analysed by various two-mode models. Although these models enable us to extract various structures underlying the asymmetric interactions among members, we are unable to extract dynamics underlying these interactions. Complex difference system models discussed in this paper enable us to describe curious dynamics of these interactions among members.

## 1 Introduction

Asymmetric relationships between objects are frequently observed in the phenomena observed in various branches of sciences. A typical example in psychology would be a one-sided affection among members of any informal group. The amount of migration from one region to another in geography is another example. These data can be arranged in matrix form. In sociomatrix its row indicates, say, rater, and its column denotes ratee. Such a data matrix is generally asymmetric. We call such a relational data matrix *asymmetric similarity matrix*. We shall hereafter abbreviate it as *ASM* between objects. Here, objects are sometimes called nodes in graph theory. Suppose that we have a set of longitudinal asymmetric similarity matrices. This type of data belongs to *two-mode three-way data*, because the rows and columns belong to the same category, i.e., objects or nodes, and time belongs to the second category. Broadly speaking this type of data can be said to be a *three-way data*.

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Naohito Chino, Emeritus Professor
Aichi-Gakuin University, Nissin-city, Aichi, Japan, e-mail: chinoe@dpc.agu.ac.jp

Two-mode three-way data including a set of longitudinal ASMs can be analysed using several statistical and/or mathematical models (e.g., Desarbo et al., 1992; Grorud et al., 1995; Okada & Imaizumi, 1997, 2004; Zielman, 1991; Zielman & Heiser, 1991). These models can be thought of as extensions of the individual differences MDS proposed by Carroll and Chang (1970) to ASMs. To be precise, these models reduce differences or changes in an ASM between objects or nodes to the individual differences. In other words, a major concern of these models can be said to obtain some *static structures* of asymmetric relationships among objects or nodes.

In contrast, there have been some models which are intended to obtain some *dynamic structures* of these asymmetric relationships among objects or nodes (e.g., Chino & Nakagawa, 1990; Gregerson & Sailer, 1993; Tobler, 1976-77; Yadohisa & Niki, 1999). For example, Chino and Nakagawa (1990) fitted a set of two-dimensional nonlinear differential equations to a set of longitudinal sociomatrices gathered by Newcomb (1961), and obtained several qualitative patterns of the trajectories of the vector fields in which members interact with each other. Here, the vector field at each point in time is estimated from the data. In other words, a major concern of this model is to obtain some dynamic structures of asymmetric relationships among members. Thus, this model can be said to be a dynamical system scaling and we call it DYNASCAL. Gregersen and Sailer (1990) examined a metamodel of two-person social systems described by a set of real two-dimensional nonlinear difference equations, and found curious chaotic behaviors. These equations include Mandelbrot's set. Tobler (1976/77) proposed a "wind" model for the interaction between geographical areas. In his model, the ASM is, for example, the amount of migration from place $i$ to place $j$. The wind is interpreted as facilitating the interaction between geographical areas in particular directions. Tobler estimates a special vector filed on a map from the data, and then decomposes it into divergence- and curl-free parts, and finally calculates the scalar and vector potentials. Yadohisa and Niki (1999) proposed a vector field representation of asymmetric proximity data, especially the scalar potential of the field.

Among these models, DYNASCAL has excellent features since it utilises qualitative theories of dynamical system, such as those of *singularities, structural stability*, and *bifurcations of vector field*. As a result, given the longitudinal AMSs between members, it draws a two-dimensional vector field on the estimated configuration of members at each time. Furthermore, it depicts singularities and several fundamental solution curves peculiar to each of the vector fields. This enables interpretation of global and local dynamical properties of the group structure at each time. However, DYNASCAL has several disadvantages, too, of which we describe four of them. Firstly, it presupposes asymmetric relationships between members but the estimated relationships are symmetric. Secondly, it might not be fully justified mathematically to administer the Procrustes rotations to the neighboring pairs of configurations. The reason for this is that DYNASCAL assumes a deterministic, nonlinear solution curve of each member in a state space as an underlying dynamics which cannot be sometimes congruent with the Procrustes rotations. Thirdly, DYNASCAL will not capture the so-called chaotic behaviors since it is restricted to a two-dimensional differential system. Fourthly, it is not possible for DYNASCAL to examine the behaviors of the system theoretically, since it merely estimates the solution curves using spline functions (Chino, 2005).

In this paper, we shall discuss complex difference system models for asymmetric interaction which were first proposed at The International Conference on Measurement and Multivariate Analysis held on May 2000 in Banfff, Canada in order to overcome those difficulties pointed out above and subsequently developed further by the author (Chino, 2000, 2001, 2002, 2003, 2005, 2006, 2014, 2015a,b).

## 2 Earlier version of the complex difference system models

The complex difference system models we proposed elsewhere (Chino, op. cit.) have several assumptions. Firstly, the *state space* in which we embed members (objects, nodes) is assumed to be a finite-dimensional Hilbert space or an indefinite-metric space. If we restrict our attention to a one-dimensional space, then an indefinite-metric space may be identified with a Hilbert space. This assumption can be justified by the *Hermitian form model* (abbreviated as *HFM*) which is underpinned by the Chino-Shiraiwa theorem (1993). In fact, in HFM any ASM, say, $\boldsymbol{S}$, is decomposed into two parts as follows:

$$\boldsymbol{S} = \frac{1}{2}(\boldsymbol{S} + \boldsymbol{S}^t) + \frac{1}{2}(\boldsymbol{S} - \boldsymbol{S}^t) = \boldsymbol{S}_s + \boldsymbol{S}_{sk}, \tag{1}$$

where $\boldsymbol{S}$ is a square asymmetric matrix of order $n$ which is the number of objects, and $\boldsymbol{S}_s$ and $\boldsymbol{S}_{sk}$ are called the symmetric part and the asymmetric part (to be precise, the skew-symmetric part), respectively. This decomposition has been used extensively in the literature (e.g., Beh & Lonbardo, 2022; Bove, 1992; Constantine & Gower, 1978; Escoufier & Grorud, 1980; Gower, 1977; Greenacre, 2000).

HFM is deduced by reinterpreting the eigenvalue problem of the Hermitian matrix $\boldsymbol{H}$, which is constructed uniquely from the observed real square asymmetric matrix $\boldsymbol{S}$, from the view point of asymmetric MDS, or, stated another way, from a geometric view point. Here, the Hermitian matrix $\boldsymbol{H}$ is simply computed as follows:

$$\boldsymbol{H} = \boldsymbol{S}_s + i\,\boldsymbol{S}_{sk}, \tag{2}$$

where $i$ is the imaginary number, that is, a square root of $-1$. Eq. (2) is nothing but the definition of Hermitian matrix. For, if $\boldsymbol{H}$ is Hermitian, then the conjugate transpose of $\boldsymbol{H}$ is $\boldsymbol{H}$ (e.g., Wilkinson, 1965). Here, it should be noticed that in general $\boldsymbol{S}^t_{sk}$ is equal to $-\boldsymbol{S}_{sk}$. $\boldsymbol{H}$ is thought of as a complexification of a real matrix $\boldsymbol{S}$, and there is a one-to-one correspondence between them. Escoufier and Grorud (1980) also utilize this equation in their asymmetric MDS. However, they do not solve the eigenvalue problem of $\boldsymbol{H}$ defined by this equation directly. Instead, they solve it by defining a real symmetric matrix of order $2n$ such that

$$\boldsymbol{H} = \begin{pmatrix} \boldsymbol{S}_s & -\boldsymbol{S}_{sk} \\ \boldsymbol{S}_{sk} & \boldsymbol{S}_s \end{pmatrix}.$$

Let us rewrite the eigenvalue problem of $\boldsymbol{H}$, i.e., $\boldsymbol{H}\,\boldsymbol{u}_j = \lambda_j\,\boldsymbol{u}_j$, as follows:

$$\boldsymbol{H} = \boldsymbol{U}_1 \boldsymbol{\Lambda}_p \boldsymbol{U}_1^*, \tag{3}$$

Here the $p \times n$ matrix $\boldsymbol{U}_1^*$ is the *conjugate transpose* of the $n \times p$ matrix $\boldsymbol{U}_1$. Of course, the $p \times p$ matrix $\boldsymbol{\Lambda}_p$ is a real diagonal matrix $\boldsymbol{\Lambda}_p = diag\,(\lambda_1, \cdots, \lambda_p)$ consisting of the non-zero eigenvalues of $\boldsymbol{H}$ arranged in descending order. $\boldsymbol{U}_1$ is consisting of $p$ eigenvectors corresponding to these non-zero eigenvalues. If we define an $n$ by $n$ matrix $\boldsymbol{U}$ which is composed of the eigenvectors associated with all the eigenvalues including zeros of $\boldsymbol{H}$ as

$$\boldsymbol{U} = \{\underbrace{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_p,}_{p}\ \underbrace{\boldsymbol{u}_{p+1}, \cdots, \boldsymbol{u}_n}_{n-p}\} = (\boldsymbol{U}_1,\ \boldsymbol{U}_2), \tag{4}$$

then $\boldsymbol{U}_1$ is the first part of the *unitary matrix* $\boldsymbol{U}$ corresponding to the non-zero eigenvalues.

Let us now rewrite (3) as

$$h_{jk} = \varphi\,(\boldsymbol{\tau}_j,\ \boldsymbol{\tau}_k) = \boldsymbol{\tau}_j\,\boldsymbol{\Lambda}_p\,\boldsymbol{\tau}_k^*, \tag{5}$$

then $\varphi\,(\boldsymbol{\tau}_j,\ \boldsymbol{\tau}_k)$ satisfies the properties of *Hermitian form* (Cristescu, 1977; Lancaster & Tismenetsky, 1985), where $\boldsymbol{\tau}_j$ is a $p$-dimensional row vector corresponding to the $j$th row of $\boldsymbol{U}_1$. Furthermore, (5) associates $h_{jk}$ with a Hermitian form.

Chino and Shiraiwa (1993) proved that $n$ objects are embedded in a finite-dimensional complex (f.d.c.) *Hilbert space* if $\boldsymbol{H}$ is positive semi-definite (p.s.d.) (or negative semi-definite (n.s.d.)), whereas they are embedded in an *indefinite metric space* if $\boldsymbol{H}$ is indefinite.

Another assumption is composed of the following basic principles of interpersonal behaviors:

1. The asymmetric sentiment relationships among members make their affinities change.
2. If a member has a positive sentiment toward another member, then he or she approaches to the target member.
3. If a member has a negative sentiment toward another member, then he or she departs from the target member.

There exist two minor principles in this family, as listed below:

1. The magnitude of change in coordinate of members is proportional to the *sine* of the difference in angles (arguments) between two members in a complex plane.

2. The magnitude of change in coordinate of members is proportional to the *norm* of the coordinate in a complex plane.

The complex difference system models were defined under the above assumptions as follows:

$$z_{j,n+1} = z_{j,n} + \sum_{m=1}^{q} \sum_{k \neq j}^{N} \boldsymbol{D}_{jk,n}^{(m)} \boldsymbol{f}^{(m)}(z_{j,n} - z_{k,n}) + z_0, \quad j = 1, 2, \ldots, N, \tag{6}$$

where

$$\boldsymbol{f}^{(m)}(\boldsymbol{z}_{j,n} - \boldsymbol{z}_{k,n}) = \left( (z_{j,n}^{(1)} - z_{k,n}^{(1)})^m, (z_{j,n}^{(2)} - z_{k,n}^{(2)})^m, \ldots, (z_{j,n}^{(p)} - z_{k,n}^{(p)})^m \right)^t, \tag{7}$$

and

$$\boldsymbol{D}_{jk,n}^{(m)} = diag\left( w_{jk,n}^{(1,m)}, w_{jk,n}^{(2,m)}, \ldots, w_{jk,n}^{(p,m)} \right), \tag{8}$$

$$w_{jk,n}^{(l,m)} = a_n^{(l,m)} r_{j,n}^{(l,m)} r_{k,n}^{(l,m)} \sin\left( \theta_{k,n}^{(l,m)} - \theta_{j,n}^{(l,m)} \right), \quad l = 1, 2, \ldots, p, \quad m = 1, 2, \ldots, q, \tag{9}$$

Here, $z_{j,n}$ denotes the coordinate vector of member $j$ at time $n$ in a $p$-dimensional *Hilbert space* or a $p$-dimensional *indefinite metric space*. Moreover, $m$ denotes the degree of the vector function $\boldsymbol{f}^{(m)}(z_{k,n} - z_{j,n})$ in (7), which is assumed to have the maximum value $q$. $z_0$ is a complex constant. Furthermore, $a_n^{(l,m)}$ is a real constant coefficient of the term $(z_{j,n}^{(l)} - z_{k,n}^{(l)})^m$, $r_{j,n}^{(l,m)}$ and $\theta_{j,n}^{(l,m)}$ are, respectively, the *norm* and the *argument* of $z_{j,n}$ at time $n$ on dimension $l$. Usually, both of $r_{j,n}^{(l,m)}$ and $\theta_{j,n}^{(l,m)}$ are independent of $m$.

At this point, we shall briefly explain how these results in (6) through (9) relate to $\boldsymbol{S}$, $\boldsymbol{H}$, $\boldsymbol{U}$ (especially, $\boldsymbol{U}_1$), and $\boldsymbol{\tau}$'s introduced previously. The matrix $\boldsymbol{S}$ in (1) consists of *observed similarities*, $s_{jk}$, between objects, and thus it is a *real matrix*. On the other hand, the matrix $\boldsymbol{H}$ in (2) consists of *hypothetical similarities*, $h_{jk}$, between objects, and it is a *complex matrix*. It is apparent that there is a one-to-one correspondence between $\boldsymbol{S}$ and $\boldsymbol{H}$.

In HFM, we decompose $\boldsymbol{H}$ into $\boldsymbol{\Lambda}_p$ and $\boldsymbol{U}_1$ which are composed of the non-zero eigenvalues and eigenvectors corresponding to these eigenvalues of $\boldsymbol{H}$, as shown in (3). According to the Chino-Shiraiwa theorem, objects are embedded in a $p$-dimensional Hilbert space if $\boldsymbol{H}$ is p.s.d., and are embedded in an indefinite metric space if $\boldsymbol{H}$ is indefinite which means that $\boldsymbol{H}$ has both positive and negative eigenvalues. In any case, $\boldsymbol{\tau}_j$ and $\boldsymbol{\tau}_k$ in (5) are $p$-dimensional row vectors corresponding to the $j$th row and $k$th row, respectively, of $\boldsymbol{U}_1$ in (3). Therefore, $\boldsymbol{\tau}_j$ and $\boldsymbol{\tau}_k$ are coordinate vectors of objects $j$ and $k$, respectively in a $p$-dimensional Hilbert space if $\boldsymbol{H}$ is p.s.d.. From (1) through (5), it is apparent that these coordinate vectors (eigenvectors) and eigenvalues explain the hypothetical similarities, $h_{jk}$, and corresponding observed asymmetric similarities, $s_{jk}$, between objects.

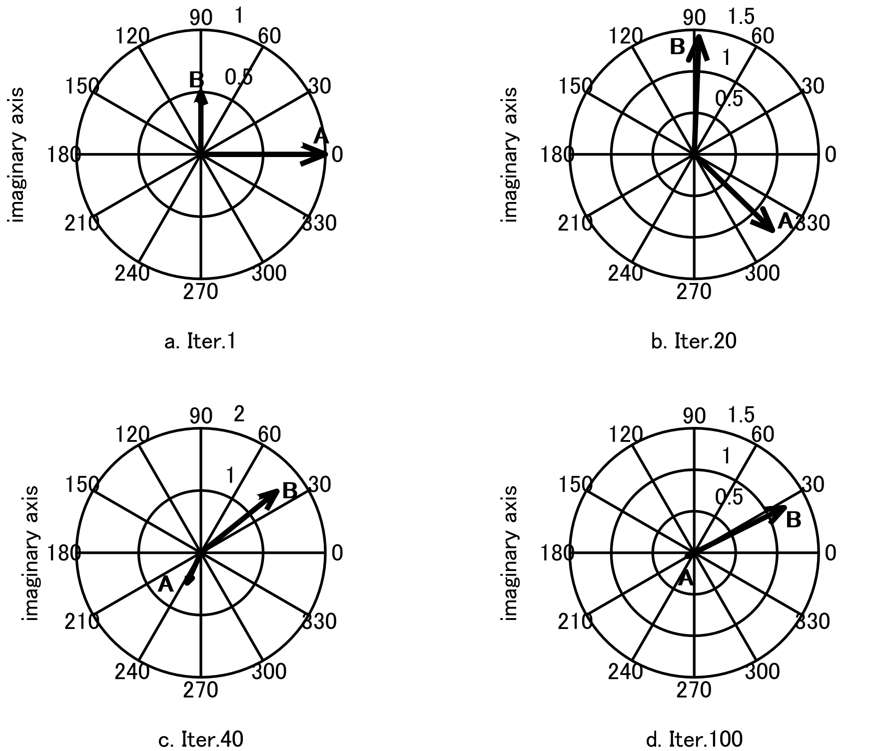


**Fig. 1** Changes in configurations of two members in a one-dimensional Hilbert space at iterations, 1, 20, 60, and 100 in a simulation study.

In our complex difference system models, we model the changes in observed asymmetric similarities, $s_{jk}$, over time. Since there is a one-to-one correspondence between $\boldsymbol{S}$ and $\boldsymbol{H}$, and since the eigenvalue problem of $\boldsymbol{H}$ gives us the complex coordinate vectors of objects in a Hilbert space if $\boldsymbol{H}$ is p.s.d., we consider these vectors as *state vectors* of objects which change over time. Here, we assume that *hypothetical asymmetric interactions* between objects exist which cause the changes in state vectors over time. The $w_{jk,m}^{(l,m)}$ are parameters concerned with these hypothetical asymmetric interactions. The $z_{j,n}$ in (6) is nothing but these state vectors at time $n$ in a $p$-dimensional Hilbert space. It should be noticed that the second right-hand side of (6) is a vector function since $\boldsymbol{D}_{jk,n}^{(m)}$ defined in (8) is a $p \times p$ diagonal matrix and $\boldsymbol{f}^{(m)}(z_{k,n} - z_{j,n})$ is a $p$-dimensional column vector defined in (7). As a result, each element of the vector function represented by the second right-hand side of (6) is a complex polynomial function of $(z_{j,n} - z_{k,n})$ whose degree is $q$. Finally, $z_0$ is a complex constant since the location of object $j$, $z_{j,n}$, in (6), which is embedded in a $p$-dimensional Hilbert space, is a complex vector.

Fig. 1 shows an example of simulations using a special case of the above difference systems in which we show changes in configurations of two members in a one-dimensional Hilbert space. This special case is written as follows for $n$-iteration:

$$\begin{cases} w_{jk,n} = a(|z_{j,n}||z_{k,n}|)^a \sin(\theta_{k,n} - \theta_{j,n}), \\ z_{j,n+1} = z_{j,n} + w_{jk,n}(z_{j,n} - z_{k,n})^2, \\ z_{k,n+1} = z_{k,n} + w_{jk,n}(z_{j,n} - z_{k,n})^2, \end{cases}$$

where $a$ is a scaling factor of the configuration which controls the domain (i.e., the coordinate at time $n$) and range (i.e., the coordinate at time $n+1$) of the configuration, and is a special case of $a_n^{(l,m)}$ in (9). In this simulation it was set equal to $1/50$. Moreover, the degree of the polynominal of $\boldsymbol{z}_{j,n} - \boldsymbol{z}_{k,n}$ in (6) was assumed to be 2, as can be seen from the above equations. Furthermore, in this case the $p$-dimensional vector $\boldsymbol{z}_{j,n}$ in (6) becomes a scalar $z_{j,n}$, because we assume here that $p = 1$. Finally, we set the initial configuration $(z_{j,1}, z_{k,1})$ of the above difference systems equal to (1, $i/2$). The reason for setting this configuration is that the skewness of the similarities between two members $j$ and $k$ is theoretically the largest of all, if the angle between two members in the complex space is $\pi/2$ (see, for example, Chino, 2020a). In Fig.1 we denote $z_{1,n}$ and $z_{2,n}$ simply by $A$ and $B$, respectively. Moreover, *time* is identified with *iteration*. Thus, for example, $A$ and $B$ in Fig.1a indicate $z_{1,1}$ and $z_{2,1}$, respectively, in the initial configuration of members. Since the angle between two members at iteration 1 is $\pi/2$, this means that member $B$ likes member $A$ very much but member $A$ does not like member $B$ at all at iteration 1. As for the interpretation of the configuration of objects in HFM, see Chino (2020a). Finally, the complex constant $z_0$ in (6) was set equal to zero.

Fig. 2 shows the changes in self-similarities of two members and those in angles over 200 iterations. In Fig. 2c one can see that the angle between two members approaches $\pi$ as the iterations increase. Fig. 3 illustrates changes in locations of two members over 200 iterations in a one-dimensional Hilbert space. In this figure, $A_1$ and $B_1$ indicate initial points of members $j(=1)$ and $k(=2)$, respectively. To be precise, coordinates of $A_1$ and $B_1$ in this complex plane are $(1, 0)$ and $(0, i/2)$, respectively.

Similarly, if we set a nonzero value to $z_0$ in (6), we can obtain more curious patterns of changes in locations of members over iteration than those in Fig. 3. However, there is a serious drawback in the complex system described by (6), (7), (8), and especially in (9). That is, the function $w_{jk,n}^{(l,m)}$ in (9) is not *holomorphic* (Chino, 2014), since both $r$ and $\theta$ are the functions of $z$ and the conjugate of $z$, i.e., $\bar{z}$. Here, *holomorphic* means *complex differentiable* (e.g., Ebeling, 2007). The complex differentiability of a complex-valued function is a natural extension of the differentiability of a real-valued function in a real space to that of a complex-valued function in a complex space. As a result, we cannot examine mathematical properties of the above different system models using complex differential calculus. The late K. Shiraiwa (personal communication, March 3, 2014), who had long been one of my colleagues, pointed out this drawback. Therefore, we have discarded (9) in our complex difference system models since then. The next section discusses a revised version of the complex difference system models which are composed of holomorphic functions.

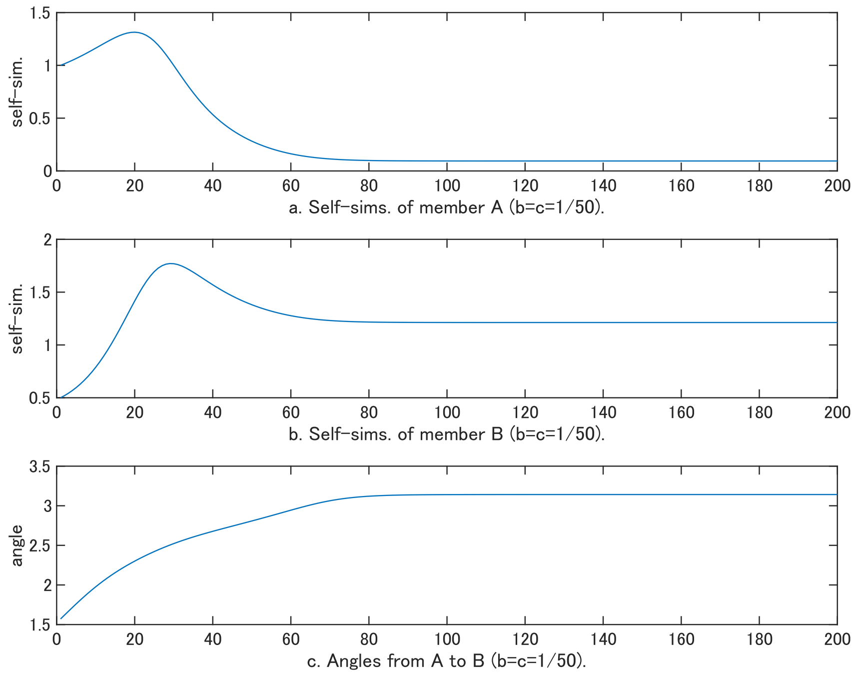


**Fig. 2** Changes in self-similarities of two members as well as the angles between them in a one-dimensional Hilbert space over 200 iterations in a simulation study.

## 3 Revised version of the complex difference system models

The revised version of the complex difference system models which are composed of holomorphic functions (Chino, 2016a,b, 2017) is nothing but a simplified version of the earlier version without (9) in the previous section. As a result, all the $w_{jk,n}^{(l,m)}$ in (8) become *complex constants*, and the corresponding minor principles in the previous section are no longer necessary. Thus, we have the following complex difference system models in a strict sense:

$$z_{j,n+1} = z_{j,n} + \sum_{m=1}^{q} \sum_{k \neq j}^{N} \boldsymbol{D}_{jk,n}^{(m)} \boldsymbol{f}^{(m)}(z_{j,n} - z_{k,n}) + z_0, \quad j = 1, 2, \ldots, N. \tag{10}$$

Here,

$$\boldsymbol{f}^{(m)}(z_{j,n} - z_{k,n}) = \left( (z_{j,n}^{(1)} - z_{k,n}^{(1)})^m, (z_{j,n}^{(2)} - z_{k,n}^{(2)})^m, \ldots, (z_{j,n}^{(p)} - z_{k,n}^{(p)})^m \right)^t, \tag{11}$$

and

$$\boldsymbol{D}_{jk,n}^{(m)} = diag\left( w_{jk,n}^{(1,m)}, w_{jk,n}^{(2,m)}, \ldots, w_{jk,n}^{(p,m)} \right). \tag{12}$$

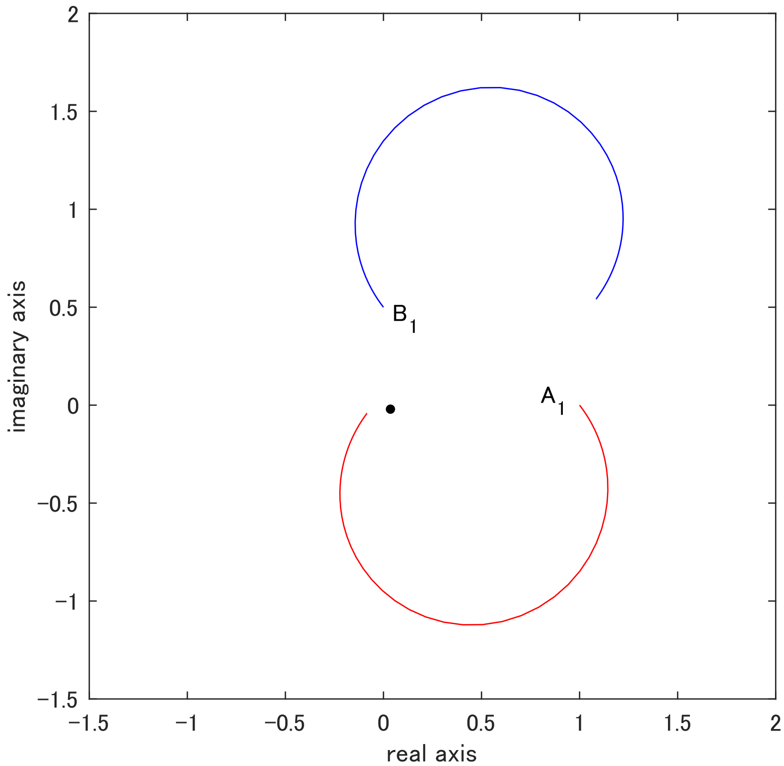


**Fig. 3** Changes in locations of two members over iterations in a one-dimensional Hilbert space.

Equations (10) through (12) are the same as (6) through (8), but no constraints are imposed on the elements of the diagonal matrix $\boldsymbol{D}^{(m)}_{jk,n}$ in (12). In other words, in the revised version we have discarded the weight constraints (9) in the earlier version. Therefore, $w^{(1,m)}_{jk,n}, w^{(2,m)}_{jk,n}, \ldots, w^{(p,m)}_{jk,n}$ are considered as free parameters in the revised version. This means that we may assume any values in these parameters.

Chino (2017) added a control term $\boldsymbol{g}(\boldsymbol{u}_{j,n})$ to the right-hand side of (10). In general, a control term in a control theory (e.g., Elaydi, 1991) is a forcing term which controls a (difference or differential) system from its outside. In (10), the control can be applied to affect directly each of the state variables $z_{1,n}, z_{2,n}, \ldots, z_{N,n}$. In this case, (10) is revised as follows:

$$z_{j,n+1} = z_{j,n} + \sum_{m=1}^{q} \sum_{k \neq j}^{N} \boldsymbol{D}^{(m)}_{jk,n} \boldsymbol{f}^{(m)}(z_{j,n} - z_{k,n}) + \boldsymbol{g}(\boldsymbol{u}_{j,n}) + z_0, \quad j = 1, 2, \ldots, N, \tag{13}$$

where $\boldsymbol{g}(\boldsymbol{u}_{j,n})$ is a *control* (e.g., Elaydi, 1999; Ott et al., 1990).

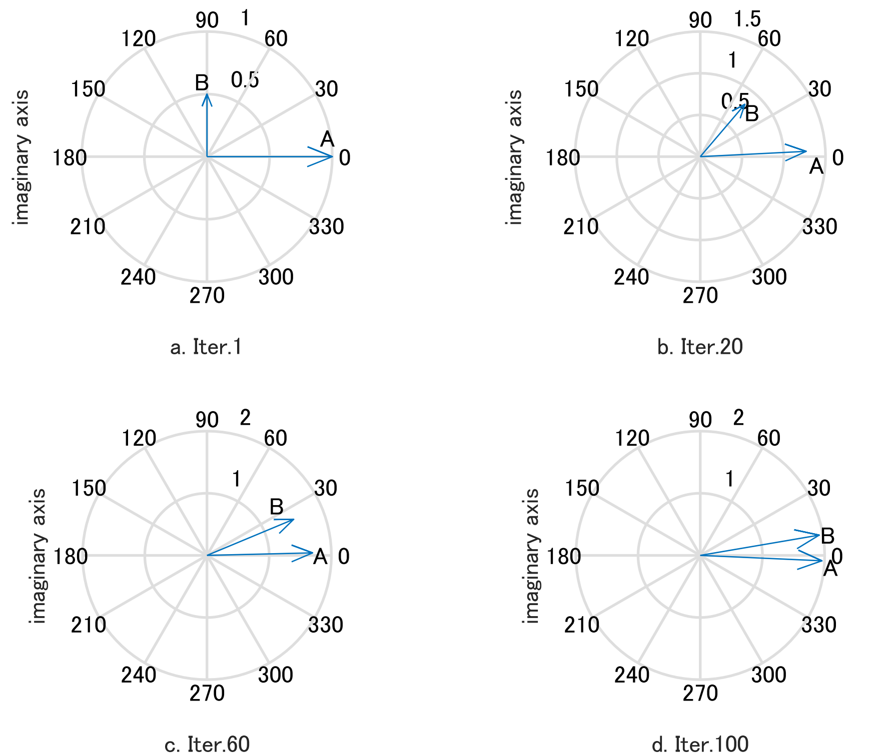


**Fig. 4** Changes in configurations of two members in a one-dimensional Hilbert space at iterations, 1, 20, 60, and 100, in another simulation study.

Moreover, we assume in the revised version that members obey only the three basic principles of interpersonal behaviors discussed in Section 2. It should be noted that we discarded the two minor principles discussed there. Fig. 4 shows another example of simulations using a special case of the above difference systems, in which we show changes in configurations of two members in a one-dimensional Hilbert space (Chino, 2016a). This case is written as follows for $n$-iteration:

$$\begin{cases} w_{jk,n} = 0.01(1+i), w_{kj,n} = -0.02(1+i), \\ z_{j,n+1} = z_{j,n} + w_{jk,n}(z_{j,n} - z_{k,n}), \\ z_{k,n+1} = z_{k,n} + w_{kj,n}(z_{k,n} - z_{j,n}), \end{cases}$$

The initial coordinates of two members, $z_{j,1}$ and $z_{k,1}$, were set equal to 1 and $i/2$, which means that the initial configuration of members is the same as that in the example shown in Section 2. However, in this example, the system is linear in contrast with the system shown in Section 2. In general, a wide class of linear difference equations can be solved explicitly and the qualitative behaviors of the solution curves in these equations are simple. However, most nonlinear difference equations cannot be solved explicitly (e.g., Cull et al., 2005; Elaydi, 1991). Moreover,

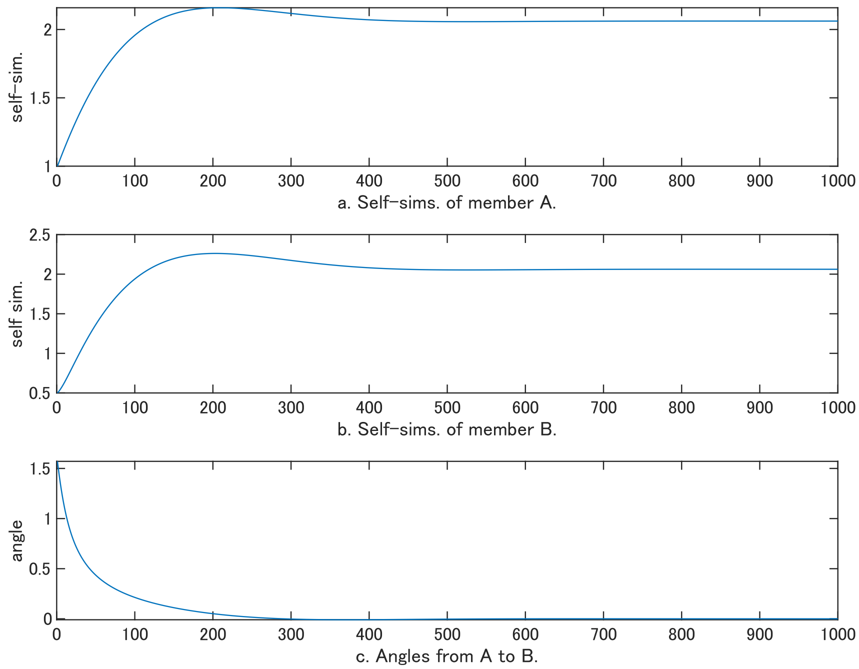


**Fig. 5** Changes in configurations of two members in a one-dimensional Hilbert space over 1000 iterations in another simulation study.

the elements of the diagonal matrix $\boldsymbol{D}_{jk,n}^{(m)}$ in (12) are assumed to be (complex) constants in marked contrast to those of the diagonal matrix in the earlier version. Note that in the earlier version the elements of the diagonal matrix vary with time $n$ according to (9). Finally, the reason why we consider here a linear system as an example of the above revised version is that we can solve this kind of linear system analytically. In fact, we can prove that, for example, the above system has a *fixed point* using a familiar method called the *Putzer algorithm* in difference equations (e.g., Cull et al., 2005; Elaydi, 1991). If we apply this algorithm to the above system, we can compute its fixed point as $2 - 0.5i$, although we shall not show its proof here because it is beyond the scope of this paper. In the following we shall check whether the fixed point of the above system approaches to this value.

Fig. 5 shows the changes in self-similarities of two members as well as angles over 1000 iterations. In Fig. 5c one can see that the angle between two members approaches 0 as iteration proceeds. Fig. 6 illustrates changes in locations of two members over 1000 iterations in a one-dimensional Hilbert space. In this figure, $A_1$ and $B_1$ indicate initial points of members $j$ and $k$, respectively, as in Fig. 3 in the previous section. As can be seen in this figure, the speed of convergence became slower and slower as locations of two members approach the fixed point. Even after 500 iterations these locations did not reach the fixed point. However, after 1000 iterations, those of members $A$ and $B$ reached $2.0 - 0.5000i$ and $2.0 - 0.5001i$, respectively. This means that two members become deeply in love with each other as iteration proceeds.

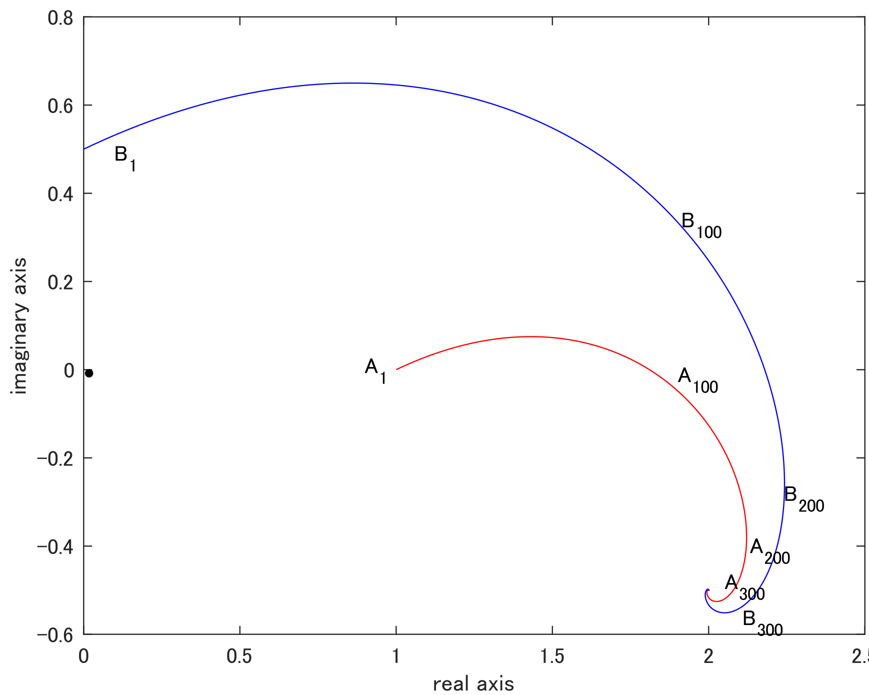


**Fig. 6** Changes in locations of two members in a one-dimensional Hilbert space over 1000 iterations in the above difference system.

In this way, we can find various patterns of dynamics which are generated by the asymmetric interactions among members. Such a job might be said to be a *classification of dynamics* generated by the complex interactions among objects. This type of classification of dynamics may be contrasted with a *classification of the static structures* among members obtained by applying a traditional two-mode three-way asymmetric MDS to a longitudinal set of asymmetric matrices.

## 4 Discussion

The complex difference system models for asymmetric interaction discussed in this paper were first proposed by the author at The International Conference on Measurement and Multivariate Analysis held on May, 2000, in Banff, Canada, and have been revised since then, as introduced in Sections 2 and 3. In these sections we have been mainly concerned with social interactions. However, asymmetric interaction can be observed ubiquitously not only in our daily lives but also *in vivo*, *in vitro*, and studies in the field, in various disciplines of science. For example, pecking order among hens and cocks is a special asymmetric interaction in ethology (e.g., Masure & Allee, 1934). Biosynthetic pathway of proteins in mammals has one-sided paths and cycles (e.g., Imai & Guarente, 2014), which can be considered as an asymmetric interaction among proteins. Weight matrix among hidden layers in neural networks represents asymmetric interactions in the brain (e.g., Goodfellow et al., 2016).

Considering these phenomena as well as the relation between weight matrix and directed graph (abbreviated as *digraph*), we have recently renamed our complex difference system models with holomorphic functions *dynamic weighted digraph* (abbreviated as *DWD)* (Chino, 2018, 2019, 2020b, 2021). Here, if a number is associated with an edge of a graph, these numbers are called *weights*, and a matrix with these numbers is called a *weight matrix*. In a digraph, the weight matrix is generally asymmetric. Therefore, in DWD asymmetric interactions are no longer restricted to social interactions. As discussed in Chino (2018), the weighted digraph in DWD is a digraph with weights specified at time $n$, which are attached to each *directed arc* (or edge, link) between *nodes* (or vertices) as well as each *loop* of the digraph. Moreover, our elementary theory of DWD assumes that the weight matrix denotes the proximity strengths among nodes at any instance of time, and that it varies as time proceeds. As a result, we obtain a set of longitudinal ASM introduced in the introductory section.

As in the complex difference system models with holomorphic functions, the state space in which we embed members (objects, nodes) is assumed to be a finite-dimensional Hilbert space or an indefinite-metric space. It should be noted here that the state space is a *hypothetical or latent space* and cannot be observed directly. Furthermore, we assume that the configuration of nodes varies according to the *mutual interactions among nodes* as time proceeds. Parameters related to these mutual interactions are specified *a priori* as certain functions of $\alpha_{jk}^{(1,m)}$, $\alpha_{jk}^{(2,m)}$, $\ldots$, $\alpha_{jk}^{(p,m)}$, described by (12) in which $w_{jk}^{(p,m)}$ are replaced by $\alpha_{jk}^{(p,m)}$.

As also pointed out in Chino (2018), the purpose of DWD is two-fold. One is *theoretical*, and the other *practical*. For the theoretical purpose, we compute the trajectories of nodes using (10), by setting an arbitrary initial configuration of nodes. Then, we recover the longitudinal digraphs associated with these similarity matrices. We can classify the patterns of changes in digraphs over time (or iteration) according to the patterns of trajectories of nodes over time (or iteration). For practical purposes, it will be possible to demonstrate the ideas above using empirical examples. For example, if we apply HFM to an observed asymmetric similarity matrix at a point in time, we can compute a $p$-dimensional configuration of members (objects, nodes). If the Hermitian matrix $\boldsymbol{H}$ computed from the observed similarity matrix is p.s.d., we can embed a $p$-dimensional configuration of members in a Hilbert space. Then, we can use the configuration thus obtained as initial values of DWD, if we assume the hypothetical complex weights in (12). Finally, if we apply (10) to the configuration of members with initial values and these hypothetical weights, we can examine various scenarios of solution curves like those in Fig. 6. These tasks remain to be done for future works. We shall go further with details of DWD in a book to be published in the near future.

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